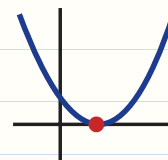


Quadratics



Syllabus topic 1.1 Quadratics · Paper 1 (Pure Mathematics 1)

For the Cambridge International AS & A Level Mathematics 9709 syllabus, exams in 2026 and 2027 (AS Level content)

Syllabus statements covered in this chapter

Syllabus ref.	What you must be able to do	Level
1.1 statement 1	Carry out the process of completing the square for a quadratic polynomial $ax^2 + bx + c$ and use a completed square form (e.g. to locate the vertex of $y = ax^2 + bx + c$ or to sketch the graph)	AS
1.1 statement 2	Find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant (e.g. to determine the number of real roots of $ax^2 + bx + c = 0$), including the term repeated root	AS
1.1 statement 3	Solve quadratic equations, and quadratic inequalities, in one unknown, by factorising, completing the square and using the formula	AS
1.1 statement 4	Solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic	AS
1.1 statement 5	Recognise and solve equations in x which are quadratic in some function of x (e.g. in x^2 , $\sqrt{x}$ or $\tan x$)	AS

What is inside

1. Concept notes, graphs and 8 worked examples
2. Exam technique; common mistakes from examiner reports
3. 20 quick-check multiple-choice questions, explained
4. 4 structured questions (Paper 1 style) with full mark schemes
5. Quick-reference sheet, "I can" checklist

How to read these notes

AS = AS Level content (Paper 1)

red = key term **green** = definition

Marks: **M1** method · **A1** accurate answer (depends on the M mark) · **B1** independent mark · **DM1** method mark that depends on an earlier M mark · **ft** = follow through

★ **Exam tip:** Paper 1 says "no marks will be given for unsupported answers from a calculator". Solve every quadratic by factorising, completing the square or the formula **with the numbers substituted**, even if your calculator can solve it.

1.1 Completing the square

A **quadratic polynomial** is $ax^2 + bx + c$ with $a \neq 0$. Its graph $y = ax^2 + bx + c$ is a **parabola**: U-shaped if $a > 0$, $\cap$ -shaped if $a < 0$.

Definition  Completing the square writes $ax^2 + bx + c$ in the form $a(x + p)^2 + q$. The graph then has its **vertex** (turning point) at $(-p, q)$ and line of symmetry $x = -p$.

When $a = 1$: halve the coefficient of x : $x^2 + bx + c = (x + \frac{b}{2})^2 - (\frac{b}{2})^2 + c$

When $a \neq 1$: take a out of the x^2 and x terms first: $ax^2 + bx + c = a[x^2 + \frac{b}{a}x] + c$, complete the square inside the bracket, then multiply back.

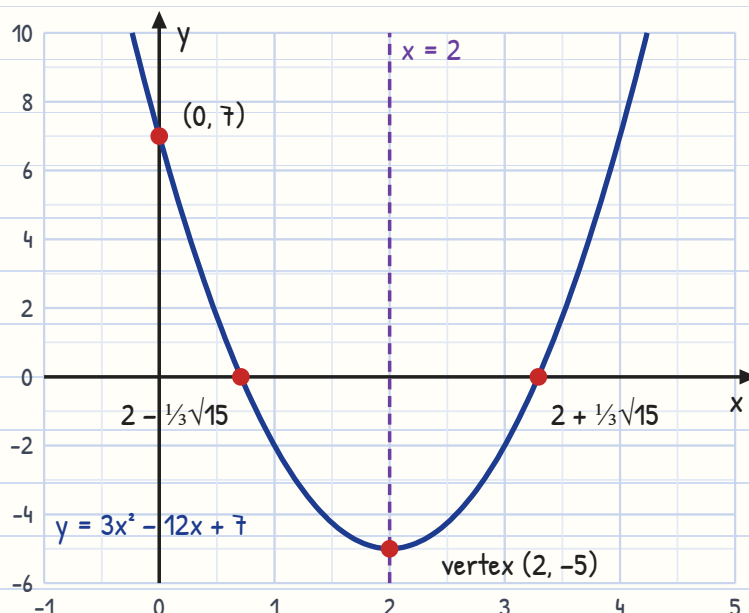
- Since $(x + p)^2 \geq 0$ for every real x : if $a > 0$ the **least value** of $a(x + p)^2 + q$ is q , when $x = -p$; if $a < 0$ the **greatest value** is q .
- For a **sketch** show the shape, the vertex, the y -intercept $(0, c)$ and any roots (where the graph meets the x -axis).

Worked example 1: completing the square with $a \neq 1$

Express $3x^2 - 12x + 7$ in the form $a(x + b)^2 + c$. Hence state the coordinates of the vertex of $y = 3x^2 - 12x + 7$ and sketch the curve.

$$3x^2 - 12x + 7 = 3[x^2 - 4x] + 7 = 3[(x - 2)^2 - 4] + 7 = 3(x - 2)^2 - 5.$$

Vertex $(2, -5)$, a minimum since $a = 3 > 0$. **y -intercept** $(0, 7)$. **Roots**: $(x - 2)^2 = \frac{5}{3}$, so $x = 2 \pm \sqrt{\frac{5}{3}} = 2 \pm \frac{\sqrt{15}}{3}$.



The sketch: U-shape, vertex $(2, -5)$, symmetric about $x = 2$.

Worked example 2: greatest value and range

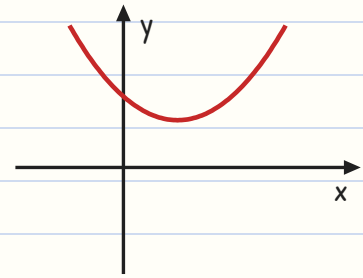
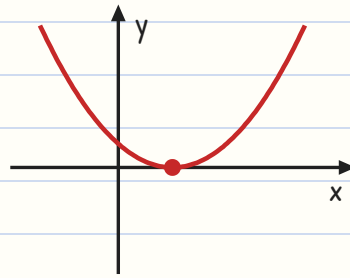
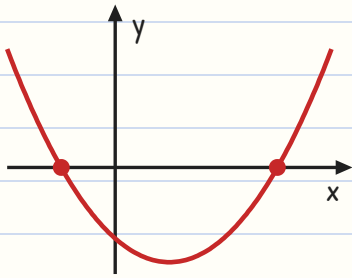
Express $5 + 4x - x^2$ in the form $a - (x - b)^2$. Hence state the greatest value of $5 + 4x - x^2$ and the range of $f(x) = 5 + 4x - x^2$ for $x \in \mathbb{R}$.

$$5 + 4x - x^2 = 5 - [x^2 - 4x] = 5 - [(x - 2)^2 - 4] = 9 - (x - 2)^2.$$

$(x - 2)^2 \geq 0$, so the greatest value is **9**, when $x = 2$. Range: $f(x) \leq 9$.

The discriminant

Definition  The **discriminant** of $ax^2 + bx + c$ is $b^2 - 4ac$. It is the expression under the square root in the quadratic formula.



$b^2 - 4ac > 0$: two distinct real roots

$b^2 - 4ac = 0$: one repeated root
(touches)

$b^2 - 4ac < 0$: no real roots

Discriminant	Roots of $ax^2 + bx + c = 0$	Graph of $y = ax^2 + bx + c$
$b^2 - 4ac > 0$	two distinct real roots	crosses the x -axis twice
$b^2 - 4ac = 0$	one repeated root (two equal roots), $x = -\frac{b}{2a}$	touches the x -axis at its vertex
$b^2 - 4ac < 0$	no real roots	does not meet the x -axis

- "Real roots" (equal or distinct) means $b^2 - 4ac \geq 0$.
- If $a > 0$ and $b^2 - 4ac < 0$, then $ax^2 + bx + c > 0$ for **all** real x (the graph is entirely above the x -axis).
- **Line and curve**: substitute the line into the curve to get a quadratic in x . Then $b^2 - 4ac > 0$: two points of intersection; $= 0$: the line is a **tangent**; < 0 : the line does not meet the curve.

Worked example 3: a repeated root

Find the values of the constant k for which the equation $x^2 + kx + k + 3 = 0$ has a repeated root.

$$b^2 - 4ac = k^2 - 4(k + 3) = k^2 - 4k - 12. \text{ Repeated root: } k^2 - 4k - 12 = 0, (k - 6)(k + 2) = 0, \text{ so } k = 6 \text{ or } k = -2.$$

(Check: $k = 6$ gives $(x + 3)^2 = 0$; $k = -2$ gives $(x - 1)^2 = 0$.)

Solving quadratic equations

1. **Factorise** (check that the brackets expand back to the original): $2x^2 + x - 6 = (2x - 3)(x + 2) = 0$, so $x = \frac{3}{2}$ or $x = -2$.
2. **Complete the square**, then square-root both sides (remember $\pm$).
3. **Formula**: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Write the formula with the values substituted.

Worked example 4: exact and 3 s.f. answers

(a) Solve $x^2 - 6x + 4 = 0$ by completing the square, giving exact answers. (b) Solve $2x^2 + 4x - 3 = 0$, giving answers to 3 significant figures.

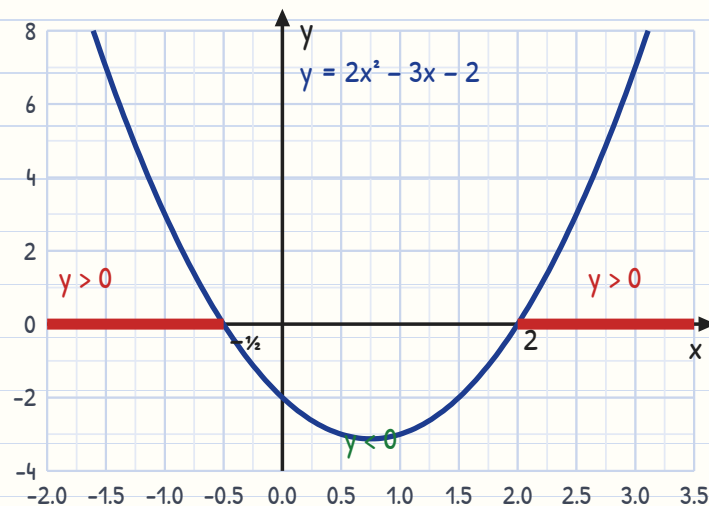
(a) $(x - 3)^2 - 9 + 4 = 0$, so $(x - 3)^2 = 5$ and $x = 3 \pm \sqrt{5}$.

(b) $x = \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)} = \frac{-4 \pm \sqrt{40}}{4}$, so $x = 0.581$ or $x = -2.58$.

Quadratic inequalities

Method: (1) rearrange to $ax^2 + bx + c > 0$ (or $<$, $\geq$, $\leq$); (2) find the **critical values** by solving $ax^2 + bx + c = 0$; (3) sketch the parabola; (4) read off where the graph is above (> 0) or below (< 0) the x -axis.

For $a > 0$ with roots $\alpha < \beta$: $(x - \alpha)(x - \beta) < 0 \iff \alpha < x < \beta$ (one interval) and $(x - \alpha)(x - \beta) > 0 \iff x < \alpha$ or $x > \beta$ (two intervals). Never write " $\beta < x < \alpha$ " or " $\alpha > x > \beta$ ".



$2x^2 - 3x - 2 > 0$ where the curve is above the x -axis (red).

Worked example 5: solving a quadratic inequality

Find the set of values of x for which $x(2x - 3) > 2$.

$2x^2 - 3x - 2 > 0$. Critical values: $(2x + 1)(x - 2) = 0$, $x = -\frac{1}{2}$ or $x = 2$. The U-shaped graph is above the axis outside the roots: $x < -\frac{1}{2}$ or $x > 2$.

Worked example 6: discriminant inequality (line meets curve)

Find the set of values of k for which the line $y = kx - 2$ meets the curve $y = x^2 + 2x + 2$ at two distinct points.

$x^2 + 2x + 2 = kx - 2 \Rightarrow x^2 + (2 - k)x + 4 = 0$. Two distinct points: $(2 - k)^2 - 16 > 0$.

Critical values: $(2 - k)^2 = 16$, $2 - k = \pm 4$, $k = -2$ or $k = 6$. The U-shaped graph of $(2 - k)^2 - 16$ against k is positive outside these: $k < -2$ or $k > 6$.

Simultaneous equations: one linear, one quadratic

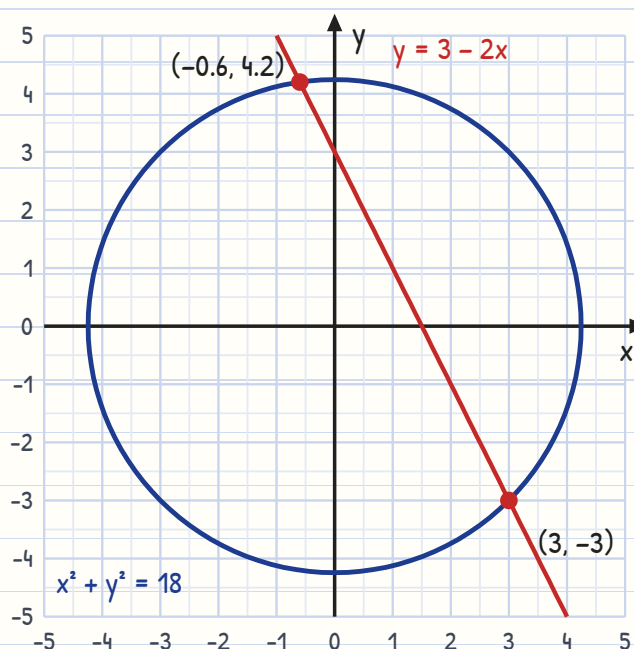
Make x or y the subject of the **linear** equation, substitute into the quadratic one, solve, then find the other variable from the **linear** equation. Give the answers in pairs. Geometrically, the solutions are the points where a line meets a curve.

Worked example 7: line and circle

Solve the simultaneous equations $2x + y = 3$ and $x^2 + y^2 = 18$.

$y = 3 - 2x$, so $x^2 + (3 - 2x)^2 = 18 \Rightarrow 5x^2 - 12x - 9 = 0 \Rightarrow (5x + 3)(x - 3) = 0$.

$x = 3: y = -3$. $x = -\frac{3}{5}: y = 3 + \frac{6}{5} = \frac{21}{5}$. Solutions $x = 3, y = -3$ and $x = -0.6, y = 4.2$.



The two solutions are the two points where the line cuts the circle.

Equations that are quadratic in a function of x

An equation of the form $a[f(x)]^2 + b f(x) + c = 0$ is a quadratic in $f(x)$. Put $u = f(x)$ (use a **new letter**, not x), solve for u , then solve $f(x) = u$ for each value, rejecting impossible ones.

Equation	Substitution	Values of u that must be rejected
$x^4 - 5x^2 + 4 = 0$	$u = x^2$	$u < 0$ (a square cannot be negative)
$x^6 - 7x^3 - 8 = 0$	$u = x^3$	none: a cube can be negative, $x = \sqrt[3]{u}$ (one value only)
$6x + \sqrt{x} - 1 = 0$	$u = \sqrt{x}$, so $x = u^2$	$u < 0$ ($\sqrt{x} \geq 0$)
$\tan^2 x = 1 + \tan x$	$u = \tan x$	none for tan; for sin or cos reject $ u > 1$
$\frac{2}{x^2} - \frac{3}{x} + 1 = 0$	$u = \frac{1}{x}$	$u = 0$ (no x gives $\frac{1}{x} = 0$)

Worked example 8: three hidden quadratics

Solve (a) $3x^4 - 11x^2 - 4 = 0$, (b) $2x - 3\sqrt{x} - 2 = 0$, (c) $2\tan^2\theta + \tan\theta - 6 = 0$ for $0^\circ \leq \theta \leq 180^\circ$.

(a) $u = x^2$: $3u^2 - 11u - 4 = (3u + 1)(u - 4) = 0$, so $u = 4$ or $u = -\frac{1}{3}$ (rejected). $x = \pm 2$.

(b) $u = \sqrt{x}$: $2u^2 - 3u - 2 = (2u + 1)(u - 2) = 0$, so $u = 2$ ($u = -\frac{1}{2}$ rejected). $x = 4$.

(c) $(2\tan\theta - 3)(\tan\theta + 2) = 0$. $\tan\theta = 1.5$: $\theta = 56.3^\circ$. $\tan\theta = -2$: $\theta = 180^\circ - 63.43^\circ = 116.6^\circ$. $\theta = 56.3^\circ$ or 116.6° .

Exam technique**How to collect every mark**

- **Show a method** for every quadratic: factors that expand to the original, the formula with numbers substituted, or completing the square. A calculator answer alone scores no marks.
- **"Hence"** means use the previous part: a completed square form gives the vertex, least value and range at once, so you need not restart with the discriminant.
- **Inequalities**: find critical values, sketch, then write the set. Keep the inequality sign correct when you divide (dividing by a negative reverses it).
- **"Set of values of k "**: the discriminant condition is itself a quadratic inequality in k : solve it like one, with a sketch.
- **Exact answers** mean surds or fractions: $3 \pm \sqrt{5}$, not 0.764 and 5.24. Otherwise give 3 significant figures, working with at least 4.
- **Check your answers make sense**: reject $x^2 < 0$ or $\sqrt{x} < 0$, and reject a value of k that stops the equation being quadratic (e.g. $k = 0$ in $kx^2 + \dots$).

Common mistakes that lose marks

- **Calculator-only solutions** of quadratics, or quoting the formula without substituting values.
- **Completing the square with $a \neq 1$** : forgetting to multiply the $-\left(\frac{b}{2a}\right)^2$ term by a : $2x^2 + 12x + 11 \neq 2(x + 3)^2 + 2$.
- **Wrong discriminant condition**: using > 0 for "does not meet", or > 0 for "real roots" (it is ≥ 0).
- **Losing a solution**: discarding the negative value of x^3 (its cube root is fine), or writing $\sqrt[3]{-8} = \pm 2$.
- **Using x as the substitution letter** for x^3 or x^2 , then forgetting to change back.
- **Algebra slips** when clearing fractions, e.g. $-\frac{1}{2x}$ becoming $-\frac{x}{2}$.
- **Inequality sets** written as one impossible statement such as $2 < x < -\frac{1}{2}$.

Quick-check multiple-choice questions

9709 has no multiple-choice paper: these 20 questions are a fast check of the skills in this chapter. Choose one answer, A, B, C or D. Answers with explanations follow the questions.

1. $x^2 + 6x + 2$ in completed square form is

A $(x + 3)^2 - 7$

B $(x + 3)^2 + 2$

C $(x + 6)^2 - 34$

D $(x - 3)^2 - 7$

13. $2x^2 - 12x + 13$ in the form $a(x + b)^2 + c$ is
 A $2(x - 3)^2 - 5$ B $2(x - 3)^2 + 4$
 C $2(x - 6)^2 - 59$ D $(2x - 3)^2 + 4$
14. The line $y = x + c$ is a tangent to the curve $y = x^2 + 3x$. The value of c is
 A 1 B -1
 C 0 D -4
15. The set of values of x for which $x^2 \geq 9$ is
 A $x \geq 3$
 B $-3 \leq x \leq 3$
 C $x \geq \pm 3$
 D $x \leq -3$ or $x \geq 3$
16. The line $y = x + 1$ meets the curve $y = x^2 - 1$ at
 A (2, 3) only
 B (1, 2) and (-2, -1)
 C (2, 3) and (-1, 0)
 D (0, 1) and (2, 3)
17. The line of symmetry of $y = 3x^2 + 12x - 1$ is
 A $x = -2$ B $x = 2$
 C $x = -4$ D $x = -6$
18. For which equation is $b^2 - 4ac = 0$?
 A $x^2 + 4x + 3 = 0$ B $x^2 + 4x + 5 = 0$
 C $x^2 + 4x - 4 = 0$ D $x^2 + 4x + 4 = 0$
19. The range of $f(x) = x^2 - 6x + 10$ for $x \in \mathbb{R}$ is
 A $f(x) \geq 10$ B $f(x) \geq 1$
 C $f(x) \geq 3$ D $f(x) > 1$
20. The solution of $\tan^2 \theta - 2 \tan \theta - 3 = 0$ with $0^\circ < \theta < 90^\circ$ is
 A 45° B 18.4°
 C 71.6° D 135°

Answers and explanations

1 A · 2 C · 3 B · 4 D · 5 B · 6 A · 7 D · 8 C · 9 A · 10 B · 11 C · 12 D · 13 A · 14 B · 15 D · 16 C · 17 A · 18 D · 19 B · 20 C

1. A. $(x + 3)^2 - 9 + 2$.
2. C. $(x - p)^2 + q$ has vertex (p, q) .
3. B. $(-3)^2 - 4(2)(5) = 9 - 40$.
4. D. $144 - 144 = 0$; in fact $(2x - 3)^2 = 0$.
5. B. $(x - 4)(x + 3) < 0$: between the roots.
6. A. $16 - (x - 3)^2$, greatest when $x = 3$.
7. D. $k^2 - 36 = 0$.

8. **C.** $(2x - 1)(x + 2) = 0$.
9. **A.** $(x - 2)^2 = 5$.
10. **B.** $(x^2 - 4)(x^2 - 9) = 0$.
11. **C.** $16 - 4k < 0$.
12. **D.** $(\sqrt{x} - 1)(\sqrt{x} - 2) = 0$, so $\sqrt{x} = 1$ or 2 .
13. **A.** $2[(x - 3)^2 - 9] + 13$.
14. **B.** $x^2 + 2x - c = 0$; $4 + 4c = 0$.
15. **D.** Outside the critical values ± 3 .
16. **C.** $x^2 - x - 2 = 0$, $x = 2$ or -1 ; $y = x + 1$.
17. **A.** $3(x + 2)^2 - 13$.
18. **D.** $16 - 16 = 0$.
19. **B.** $(x - 3)^2 + 1 \geq 1$, and the value 1 is reached at $x = 3$.
20. **C.** $(\tan \theta - 3)(\tan \theta + 1) = 0$; $\tan \theta = -1$ has no solution in range.

Exam-style structured questions (Paper 1 style)

Answer all parts. Show all necessary working: no marks are given for unsupported answers from a calculator. Total: 28 marks.

Question 1 [8 marks]

The function f is defined by $f(x) = 2x^2 + 12x + 11$ for $x \in \mathbb{R}$.

- (a) Express $f(x)$ in the form $a(x + b)^2 + c$, where a , b and c are constants. [3]
- (b) State the coordinates of the vertex of the graph of $y = f(x)$. [1]
- (c) Solve the equation $f(x) = 0$, giving your answers in exact form. [2]
- (d) Find the set of values of x for which $f(x) > 11$. [2]

Mark scheme

- (a) $2[(x + 3)^2 - 9] + 11$ or $a = 2$ seen **BI**; $b = 3$ **BI**; $c = -7$, giving $2(x + 3)^2 - 7$ **BI**
- (b) $(-3, -7)$ **BI ft** (ft their $(-b, c)$)
- (c) $(x + 3)^2 = \frac{7}{2}$ and square root taken **MI**; $x = -3 \pm \sqrt{\frac{7}{2}}$ or $-3 \pm \frac{\sqrt{14}}{2}$ **AI**
- (d) $2x^2 + 12x > 0$, critical values $x = 0$ and $x = -6$ **MI**; $x < -6$ or $x > 0$ **AI**

Question 2 [8 marks]

A curve has equation $y = x^2 + 3x - 2$ and a line has equation $y = kx - 6$, where k is a constant.

- (a) Find the set of values of k for which the line does not meet the curve. [5]
- (b) For the greater of the two critical values of k found in (a), the line is a tangent to the curve. Find the coordinates of the point where the line touches the curve. [3]

Mark scheme

- (a) $x^2 + 3x - 2 = kx - 6$ **MI**; $x^2 + (3 - k)x + 4 = 0$ **AI**; $(3 - k)^2 - 16$ with < 0 (or critical values from $= 0$) **MI**; critical values $k = -1$ and $k = 7$ **AI**; $-1 < k < 7$ **AI**
- (b) $k = 7$: $x^2 - 4x + 4 = 0$ and solved **MI**; $x = 2$ **AI**; $y = 8$, so $(2, 8)$ **AI**

Question 3 [5 marks]Solve the simultaneous equations $x + 2y = 7$ and $x^2 + 3xy = 10$.

[5]

Mark scheme

- $x = 7 - 2y$ substituted: $(7 - 2y)^2 + 3y(7 - 2y) = 10$ (M1)
- $2y^2 + 7y - 39 = 0$ (A1)
- $(2y + 13)(y - 3) = 0$ or formula with values (M1)
- $y = 3$ or $y = -\frac{13}{2}$ (A1)
- $x = 1, y = 3$ and $x = 20, y = -6.5$ (A1) (substituting for y instead leads to $x^2 - 21x + 20 = 0$; mark equivalently)

Question 4 [7 marks](a) Solve the equation $4x^4 - 37x^2 + 9 = 0$.

[4]

(b) Solve the equation $6x - 7\sqrt{x} - 3 = 0$.

[3]

Mark scheme

- (a) $u = x^2$: $4u^2 - 37u + 9 = 0$ (B1); $(4u - 1)(u - 9) = 0$ (M1); $x^2 = \frac{1}{4}$ or $x^2 = 9$ (A1); $x = \pm\frac{1}{2}, \pm 3$ (A1)
- (b) $u = \sqrt{x}$: $6u^2 - 7u - 3 = (3u + 1)(2u - 3) = 0$ (M1); $\sqrt{x} = \frac{3}{2}$, with $\sqrt{x} = -\frac{1}{3}$ rejected (A1); $x = \frac{9}{4}$ (A1)

Quick-reference sheet

Item	Key facts
completed square	$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$; vertex $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$; least value if $a > 0$, greatest if $a < 0$
discriminant	$b^2 - 4ac$: > 0 two distinct real roots; $= 0$ repeated root; < 0 no real roots; real roots $\iff \geq 0$
formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (show the values substituted)
inequalities	critical values, sketch; for $a > 0$: < 0 between the roots, > 0 outside them (two separate inequalities)
line and curve	substitute, form a quadratic; tangent $\iff b^2 - 4ac = 0$; no intersection $\iff b^2 - 4ac < 0$
simultaneous	rearrange the linear equation, substitute, solve, pair the answers
hidden quadratics	$u = x^2, x^3, \sqrt{x}, \frac{1}{x}, \tan x \dots$; reject impossible values of u

"I can" checklist

- ☐ complete the square for $ax^2 + bx + c$, including $a \neq 1$ and $a < 0$, and use it to find the vertex, the least or greatest value and to sketch the graph
- ☐ find the discriminant and use it to decide the number of real roots, including a repeated root, and to find unknown constants
- ☐ solve quadratic equations by factorising, completing the square and the formula, giving exact or 3 s.f. answers
- ☐ solve quadratic inequalities and write the solution set correctly
- ☐ solve a linear and a quadratic equation simultaneously by substitution
- ☐ recognise and solve equations that are quadratic in $x^2, \sqrt{x}, \frac{1}{x}$ or $\tan x$, rejecting impossible values